International Journal of Theoretical Physics, Vol. 34, No. 6, 1995

The q-Deformed Cartesian *osp*(1/2) Algebra

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Received October 6, 1994

The q-deformed Cartesian *osp*(1/2) algebra is introduced and its matrix representation is investigated.

1. INTRODUCTION

The quantum Yang–Baxter equation is known to play an important role in diverse problems in theoretical physics. These involve exactly soluble models in statistical mechanics (Baxter, 1982) and quantum integrable model field theory (Faddeev, 1981; Bogoliubov *et al.*, 1985; Bullough *et al.*, 1988; Sklynin, 1982; Kulish and Reshetikhin, 1983; de Vega *et al.*, 1984). Just as the Jacobi identity endows a Lie algebra with an associativity condition, the quantum Yang–Baxter equation plays a similar role for a new type of algebraic structure that is a generalization of Lie algebra.

This structure is sometimes described as the q-deformation of a Lie algebra. From a mathematical point of view, it is a noncommutative Hopf algebra, but in the context of quantum integrable models, it is called a quantum group (Doebner and Henning, 1990; Bullough *et al.*, 1990). The structure and representation of quantum groups have been developed extensively by Jimbo (1985), Drinfel'd (1986), and Faddeev (1984). The deformed analog of the harmonic oscillator has already been studied (Kuryshkin, 1980). The representation theory of the q-deformation of the Lie algebra $su(2) \rightarrow su_q(2)$ has been extensively investigated (Biedenharn, 1989; Macfahlane, 1989; Kulish and Damaskinsky, 1990; Kulish, 1981; Gerdjikov *et al.*, 1984). These papers are concerned with the q-deformed boson satisfying the deformed Heisenberg–Weyl algebra, which we call "q-bosons." Recently some physi-

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cists have considered the generalization of the standard q-deformed algebras (Polychronakos, 1990; Rocek, 1991; Daskaloyannis, 1991; Chung *et al.*, 1993; Chung, 1994).

Some physicists have considered the supersymmetric extension of the bosonic quantum group. One of the simplest examples is to obtain a supersymmetric extension of $su_q(2)$ algebra. This extended algebra is usually called $osp_q(1/2)$ algebra and much has been accomplished in investigating the properties of this algebra (Saleur, 1990; Lukierski and Nowicki, 1992; Floreanini *et al.*, 1990).

In this paper we obtain the Cartesian deformation of $osp(1/2)$ algebra and investigate the representation of the algebra. Its bosonic case is discussed in Witten (1990), Fairlie (1990), Curlie and Zachos (1990), and Zhedanov (1992).

2. q-DEFORMED CARTESIAN *osp*(1/2) ALGEBRA

In this section we discuss the q-deformed Cartesian $osp(1/2)$ algebra. Let us start with the following form of the q-deformed Cartesian $osp(1/2)$ algebra:

$$\begin{aligned} \{V_-, V_+\}_q &= H \\ \{V_-, V_-\}_q &= J_- \\ \{V_+, V_+\}_q &= J_+ \\ [H, V_+]_q &= V_+ \\ [V_-, H]_q &= V_- \end{aligned} \tag{1}$$

where the qummutator and antiqummutator are defined as

$$[A, B]_q = AB - qBA, \qquad \{A, B\}_q = AB + qBA$$

From equations (1), especially the second and third equations, one can obtain the remaining commutation relations of the q-deformed Cartesian $osp(1/2)$ algebra, which are given by

$$\begin{aligned} [J_\pm, V_\pm] &= 0 \\ [V_-, J_+]_{q^2} &= [2]V_+ \\ [J_-, V_+]_{q^2} &= [2]V_- \\ [H, J_+]_{q^2} &= [2]J_+ \\ [J_-, H]_{q^2} &= [2]J_- \\ [J_-, J_+]_{q^4} &= [2]^2(qH + (1 - q)V_-V_+) \end{aligned} \tag{2}$$

A glance at the last equation indicates that the qummutator of the two even generators (J_+, J_-) cannot be written in terms of the function in H only. However, when the deformation parameter q goes over to 1, the left-hand side of the last equation of equations (2) becomes an ordinary commutator, which is given in terms of the function in H only.

3. MATRIX REPRESENTATION

Now we will obtain the matrix representation of the algebra (1) by assuming that H is diagonal and V_+ (V_-) are super (sub) diagonal,

$$(H)_{ij} = h_i \delta_{ij}$$

$$(V_+)_{ij} = v_i \delta_{i+1,j}$$

$$(V_-)_{ij} = v_j \delta_{i,j+1} \tag{3}$$

where the last relation is obtained from the second relation by using the fact that V_- is Hermitian conjugate to V_+. Then the forth relation of equations (1) determines h_i through the recurrence relation

$$h_i - qh_{i+1} = 1 \qquad (i = 1, 2, \ldots, n - 1) \tag{4}$$

The relation (4) is easily solved and the solution is given by

$$h_i = q^{n-i}h_n + \frac{q^{-1}(q^{n-i} - 1)}{1 - q^{-1}} \tag{5}$$

where h_n will be determined by using the another relation of equations (1). Using the first relation of equations (1), we get

$$v_{i-1}^2 + qv_i^2 = h_i \quad (i = 1, 2, \ldots, n); \qquad v_0 = 0, \qquad v_n = 0 \tag{6}$$

Using equation (5) and checking the consistency of equation (6), we obtain the following identity:

$$\sum_{k=1}^{n} (-q^{-1})^{n-k} h_k = 0 \tag{7}$$

This relation and equation (5) fix the concrete form of the value of h_n. Inserting equation (5) into equation (7) produces

$$h_n \frac{(-)^n - 1}{2} = \frac{1}{1 - q} \frac{(-)^n - 1}{2} + \frac{1}{1 - q} \frac{q + (-q^{-1})^{n-1}}{1 + q} \tag{8}$$

For the case that n is even, equation (8) does not give the solution for h_n; so we assume that n is odd. Then we have

$$h_n = \frac{1}{1 - q} + \frac{q + q^{-n+1}}{(q - 1)(q + 1)} \tag{9}$$

Inserting equation (9) into equation (5), we obtain

$$h_i = \frac{1}{1-q}\left[1 - \frac{q}{1+q}(q^n + 1)q^{-i}\right] \tag{10}$$

Using equation (6), we can perform the partial summations and obtain a closed expression for v_i given by

$$v_i = \left\{\frac{1}{1-q^2}\left[1 - (-q^{-1})^i + \frac{(-)^i - 1}{2}(q^n + 1)q^{-i}\right]\right\}^{1/2} \tag{11}$$

For the $n = 3$ case we can obtain the explicit matrix representation as follows:

$$H = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 - q^{-1} & 0 \\ 0 & 0 & -q^{-2} \end{pmatrix}$$

$$V_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & iq^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -iq^{-1} & 0 \end{pmatrix} \tag{12}$$

where we assumed that q is real.

4. CONCLUSION

In this paper we constructed the q-deformed Cartesian deformation of the $osp(1/2)$ algebra and obtained the matrix representation of the algebra. In particular we found that the odd-dimensional representation is only allowed from the recurrence relation.

ACKNOWLEDGMENTS

This paper was supported by the Non-Directed Research Fund, Korea Research Foundation, 1994. The present studies were also supported in part by Basic Science Research Program, Ministry of Education, 1994 (BSRI-94-2413).

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